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REGULAR PARTIALLY INVARIANT SUBMODELS OF THE EQUATIONS OF GAS DYNAMICS†

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All regular submodels of the system of equations of gas dynamics with an equation of state of general form are described. The submodels are classified according to types determined by the rank and defect. Classification tables with comments are presented, and some characteristic examples are given. © 1997 Elsevier Science Ltd. All rights reserved.

The aim of the "PODMODELI" ("submodels") program [1] is to exhaust all possibilities arising from the symmetry of differential equations in order to construct submodels (systems of equations of lower dimension) describing classes of exact solutions of the original equations. In the present paper, within the framework of this program, we describe all (apart from similarity) regular submodels of the system of equations of gas dynamics with an equation of state of general form. Regular submodels [2] can be distinguished by the property that their independent variables are functions of the original independent variables only.

1. GENERAL NOTIONS

Consider a system E of differential equations with n independent variables $x = (x^1, \ldots, x^n)$ and m required functions $u = (u^1, \ldots, u^m)$ of the variables x. Suppose that E admits of a local Lie group H of transformations of the space R^{n+m} (x, u) with *universal invariant* $I = (I^1, \ldots, I^l)$.

Definition 1. The system of equations $E|_M$ obtained by restricting E to an invariant submanifold M of H of dimension $n + \delta$ in R^{n+m} (x, u) and dimension σ in the space of invariants $R^l(I)$, where $\sigma \geqslant 0$ and $0 \leqslant \delta < m$, is called an H-submodel of type of (σ, δ) of E. We say that σ is the *rank* and δ is the *defect* of the H-model. The solutions of the system $E|_M$ are called *partially invariant solutions* of rank σ and defect δ or, briefly, $H(\sigma, \delta)$-*solutions*.

If such an M exists, then the components of I can always be chosen in such a way that the following relationships with $u = (u', u'')$, $I = (I', I'')$, where $u' = (u^1, \ldots, u^{m-\delta})$, $I' = (I^1, \ldots, I^{m-\delta})$, are satisfied (gr stands for "general rank")

$$\partial I''/\partial u' = 0, \ \operatorname{gr}\|\partial I'/\partial u'\| = m-\delta, \ \operatorname{gr}\|\partial I''/\partial(x,u'')\| = \sigma \tag{1.1}$$

$$\sigma = l - m + \delta \tag{1.2}$$

Then, if we put

$$v = I'(x,u), \quad y = I''(x,u'') \tag{1.3}$$

the equations of M can be written as

$$M\colon v = V(y) \tag{1.4}$$

Equalities (1.3) define a *representation* of the $H(\sigma, \delta)$-solutions in terms of the invariants of H. The equations of the submodel $H(\sigma, \delta)$ can be obtained by substituting this representation into the equations E. As a result, system E splits into two subsystems: the *invariant subsystem* E/H for the unknown functions $V(y)$ and, in addition, a subsystem Π for the "superfluous" functions (SF) $u''(x)$, which is an overdetermined system, in general. If Π is inconsistent, the set of $H(\sigma, \delta)$-solutions is empty. Therefore the problem of finding $H(\sigma, \delta)$-solutions rests in the first place on the study of the consistency of the equations of Π (*reducing* Π *to involution*).

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Definition 2. The number

$$\mu = \mathrm{gr}\|\partial I'' / \partial u''\|;$$

is called the *measure of irregularity* of an $H(\sigma, \delta)$-submodel. If $\mu = 0$, the $H(\sigma, \delta)$-submodel is called *regular*, and if $\mu > 0$, it is called *irregular*.

Significant differences between regular and irregular submodels are described in [2]. In particular, for regular solutions the invariant independent variables y in (1.4) in the subsystem E/H depend only on the original independent variables, which makes it much simpler to reduce the subsystem Π to involution.

In applications one usually uses not the group H itself, but its *Lie algebra of operators with basis*

$$X_\alpha = \xi^i_\alpha(x,u)\partial_{x^i} + \eta^k_\alpha(x,u)\partial_{u^k} \quad (\alpha = 1,\dots,r) \tag{1.5}$$

Then l is determined by the general rank of the matrix formed by the coordinates of the operators in (1.5)

$$r_* = \mathrm{gr}\|\xi^i_\alpha(x,u),\quad \eta^k_\alpha(x,u)\|$$

namely, $l = n + m - r_*$. Substitution into (1.2) yields the formula

$$\sigma = \delta + n - r_* \tag{1.6}$$

which determines the rank of σ if the defect δ is given.

As is well known [3], possible values of δ satisfy the inequalities

$$\max\{r_* - n, 0\} \leqslant \delta \leqslant \min\{r_* - 1, m - 1\} \tag{1.7}$$

From (1.7) and (1.6) it follows that the number of different types (σ, δ) is equal to nm.

Definition 3. The $H(\sigma, 0)$-solutions are called *invariant H-solutions* of rank σ ($\sigma < n$ is always satisfied).

For invariant H-solutions in (1.3) we have $y = I''(x)$, i.e. all invariant H-solutions are regular. For such solutions the submodel $E|_M$ consists of one invariant subsystem E/H, the subsystem Π is empty, and there is no problem of reducing to involution.

If $\delta > 0$, the reduction of Π to involution can be a branching process, giving *various classes* of $H(\sigma, \delta)$-solutions. Some of these classes may turn out to be $H_1(\sigma_1, \delta_1)$-solutions for a *subgroup* $H_1 \subset H$. As is well known [3], it is always true that $\sigma_1 \geqslant \sigma$, $\delta_1 \geqslant \delta$ in this case. The introduction of the next notion below rests on the fact that the classes of $H(\sigma, \delta)$-solutions of the given rank having smaller defect or of smaller rank having the given defect are easier to describe and to study.

Definition 4. If a class of $H(\sigma, \delta)$-solutions is a class of $H_1(\sigma_1, \delta_1)$-solutions with *subgroup* $H_1 \subset H$ and with

$$\sigma_1 = \sigma, \quad \delta_1 < \delta \tag{1.8}$$

then it is said that *reduction* of $H(\sigma, \delta)$-solutions towards a *smaller defect* occurs. Conversely, if a class of $H(\sigma, \delta)$-solutions is a class of $H_2(\sigma_2, \delta_2)$-solutions with *subgroup* $H_2 \supset H$, where

$$\sigma_2 < \sigma, \quad \delta_2 = \delta \tag{1.9}$$

then it is said that *inverse reduction* of $H(\sigma, \delta)$-solutions to *lower rank* occurs.

Reduction to an invariant solution is particularly often encountered. Some sufficient conditions for such reduction, making it possible to predict it from the structural properties of the subsystem Π, are presented in [3].

2. THE EQUATIONS OF GAS DYNAMICS

We consider the following system E in a nine-dimensional *base space* $R^9(t, \mathbf{x}, \mathbf{u}, \rho, S)$ with independent variables t (time), $\mathbf{x} = (x, y, z)$ (Cartesian coordinates in R^3), and unknown variables $\mathbf{u} = (u, v, w)$ (velocity vector), ρ (density), and S (entropy)

$$\rho D\mathbf{u} + \nabla p = 0, \quad D\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad DS = 0, \quad p = F(\rho, S) \tag{2.1}$$

Here $D = \partial_t + \mathbf{u} \cdot \nabla$ and $\nabla = (\partial_x, \partial_y, \partial_z)$. The pressure p is determined by the *equation of state* (the last equation in (2.1)). $F(\rho, S)$ is assumed to be a given smooth function of general form that satisfies the inequalities $F_\rho = c^2 > 0$, where c is the velocity of sound and $F_S > 0$.

We know [3] that (2.1) admits of an 11-parameter local Lie group G_{11} of transformations of R^9. The Lie algebra L_{11} of this group has the following basis of operators of the form (1.5)

$$\begin{aligned}
&X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z \\
&X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w \\
&X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v \\
&X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w \\
&X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u \\
&X_{10} = \partial_t, \quad X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z
\end{aligned} \tag{2.2}$$

The *normalized optimal system of subalgebras* ΘL_{11} is given in [1]. It consists of 220 representations, each of which is a potential product of $H(\sigma, \delta)$-submodels.

In what follows these representations will be denoted by $L_{r,i}$, where r is the dimension of the subalgebra and i is the consecutive number of a subalgebra of given dimension according to the table for ΘL_{11}.

3. TYPES OF SUBMODELS

In (2.1) $n = 1$ and $m = 5$. Therefore, 20 types of $H(\sigma, \delta)$-submodels are a priori possible for (2.1). These types can be found from (1.6) and (1.7).

The results presented in Table 1 contain the initial information on the number of different submodels of each type. Here it is taken into account that submodels are determined not only by their type, but

Table 1

σ	δ	l	N	N_{reg}	Remarks
3	0	8	13 (1)	13 (1)	invariant
2	0	7	26 (2)	26 (2)	invariant
3	1		26 (2)	1 (2)	
1	0		38 (3)	38 (3)	invariant
2	1	6	51 (3)	12 (3)	
3	2		47 (3)	–	
0	0		5 (4)	5 (4)	invariant isobaric
1	1	5	46 (4) + 1 (5)	29 (4)	
2	2		47 (4) + 1 (5)	1 (4)	
3	3		47 (4) + 2 (5)	–	
0	1		22 (5) + 2 (6)	22 (5)	partial isobaric
1	2	4	35 (5) + 13 (6)	9 (5) + 1 (6)	
2	3		35 (5) + 8 (6)	–	
3	4		35 (5) + 8 (6)	–	
0	2		13 (6) + 10 (7)	13 (6) + 10 (7)	partial isobaric
1	3	3	1 (6) + 8 (7)	1 (7)	barochronic
2	4		1 (6) + 5 (7)	–	
0	3	2	1 (≥7)	1 (≥7)	isobaric
1	4		1 (7)	–	isentropic and barotropic
0	4	1	–	–	none

that they also depend on a specific representation of the Lie algebra L_{11} in terms of the operators (2.2) in R^9. For example, from (2.2) one can see immediately that ρ, S, and p are invariants of any subgroup in G_{11}.

The type (σ, δ) is shown in the first two columns of Table 1. In the third column we list possible dimensions l of the space of invariants. The fourth column provides information on the number N of subalgebras from the optimal system ΘL_{11} which can generate different models. Here N is represented as a sum $N_1(r_1) + N_2(r_2)$, where r_k is the dimensions of a subalgebra, and $N_k(r_k)$ is the number of subalgebras of dimension r_k. In the fifth column, using the same notation, we give the number N_{reg} of subalgebras generating different *regular* submodels. That no submodel of the given type exists is indicated by a dash. The sixth column lists specified classes of solutions.

Among the gas motions described by (2.1) one can distinguish the classes of motions mentioned below, which are frequently encountered in the classification of submodels.

Type (1.4). *Isentropic motions*, S = const. System (2.1) reduces to

$$\rho D\mathbf{u} + F'(\rho)\nabla\rho = 0, \quad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad p = F(\rho) \tag{3.1}$$

with given function $F(\rho)$.

Type (1.4). *Barotropic motions*, $p = P(\rho)$. Such a motion is either isentropic when $P(\rho) = F(\rho)$, or has variable entropy. In the latter case (2.1) can be reduced to

$$D\mathbf{u} + \nabla e = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \quad De = 0 \tag{3.2}$$

with *specific enthalpy* $e = e(\rho)$, in terms of which the pressure can be expressed by the formula $p = \int \rho e(\rho) d\rho$.

Type (1.3). *Barochronic motions*, $p = p(t)$, $\rho = \rho(t)$. Such motions are isentropic. System (2.1) reduces to

$$D\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{u} = -\rho'/\rho, \quad p(t) = F(\rho(t)) \tag{3.3}$$

with given function $F(\rho)$.

Type (0.3). *Isobaric motions*, p = const. System (2.1) can be reduced to the following

$$D\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{u} = 0, \quad D\rho = 0, \quad F(\rho, S) = \mathrm{const} \tag{3.4}$$

with given function $F(\rho, S)$, which determines $\rho(S)$.

The systems of equations (3.2)–(3.4) are overdetermined. For (3.4) an expression for the general solution in terms of arbitrary functions is known [4]. System (3.3) can easily be reduced to involution, but no general solution has been constructed. Also, the problem of reducing (3.2) to involution remains unsolved.

The fact that ρ and S are invariants of any subalgebra in ΘL_{11} has a considerable effect on the structure of a submodel. In particular, submodels of type (0.4) are impossible, since the rank $\sigma = 0$ means that one must put ρ = const and S = const, and $\delta = 4$ means that there must be 4 SF's, while only three, namely, u, v and w remain to be found. For the same reason, all submodels of type (0, δ) describe isobaric motions. All solutions of type (0.0) (including the constant solution) are contained in the class of solutions of type (1.0) generated by the subalgebra $L_{3,33}$ with basis X_2, X_3, X_{10} and having the form

$$u = 0, \quad v = v(x), \quad w = w(x), \quad \rho = \rho(x), \quad p = \mathrm{const}$$

All invariant submodels (of type σ, 0) are described separately and are not presented here (type (3.0) was published in [1]). Thus, below we describe only naturally possible regular submodels of type (3.1) and (2.2), and series of regular submodels of type (2.1), (1.1) and (1.2). One model of type (1.2) is distinguished. It can be called *canonical*, since it also arises in a number of other submodels.

In what follows we shall write briefly $(u, v, \ldots)|(t, x, \ldots)$ to show that $u, v, \ldots$ depend on $t, x, \ldots$.

4. CANONICAL SUBMODEL OF TYPE (1.2)

It describes a two-dimensional version of barochronic motions and is generated by the subalgebras $L_{5,17}$, $L_{5,37}$, $L^{00}_{6,8}$. They have the same system of invariants: t, u, ρ, S with SFs u, w. The solutions can be represented as

$$(u,\rho,S)|t; \quad (v,w)|(t,x,y,z)$$

From (3.3) it follows that $u'(t) = 0$, i.e. u = const. By a Galilean translation in x one can ensure that $u = 0$ and then Eqs (3.3) reduce to

$$v_t + vv_y + wv_z = 0, \quad w_t + vw_y + ww_z = 0 \tag{4.1}$$

$$v_y + w_z = 2h \tag{4.2}$$

$h = h(t)$ being a function to be determined, in terms of which $\rho = \rho(t)$ can be found from the equation $\rho' = -2h\rho$.

The consistency conditions for system (4.1), (4.2) have the form

$$v_y w_z - v_z w_y = k \tag{4.3}$$

$$k = h' + 2h^2, \quad k' + 2hk = 0 \tag{4.4}$$

By these conditions the whole system (4.1)–(4.4) is in involution. This system is interesting in that its general solution can be found in terms of arbitrary functions.

By means of the substitution

$$z = Z(t,y,v), \quad w = W(t,y,v) \tag{4.5}$$

the system can be linearized, taking the form

$$W_v = Z_y + 2hZ_v, \quad W_y = -kZ_v \tag{4.6}$$

$$W_t + vW_y = 0, \quad Z_t + vZ_y = W \tag{4.7}$$

Subsystem (4.6) can be integrated as a system of equations with constant coefficients. The form of the solution depends on the discriminant $d = h^2 - k$: system (4.6) is hyperbolic when $d > 0$, elliptic when $d < 0$, and parabolic when $d = 0$. The functions $h(t)$ and $k(t)$ are easily found from the solution of (4.4) and turn out to be rational functions of t. Subsystem (4.7) reduces to a system of ordinary differential equations and can also be integrated explicitly. Finally, the solution can be determined implicitly from (4.5) with the known functions Z and W.

5. REGULAR SUBMODEL OF TYPE (3.1)

It is generated by the subalgebra $L_{2,26}$ with basis $X_1, X_4, t, y, z, v, w, \rho, S$ are invariants and u is an SF. The solution can be represented as

$$u = u(t,x,y,z); \quad (v,w,\rho,S)|(t,y,z)$$

System (2.1) takes the form

$$D'u + uu_x = 0, \quad \rho D'v + p_y = 0, \quad \rho D'w + p_z = 0 \tag{5.1}$$

$$D'\rho + \rho(u_x + v_y + w_z) = 0, \quad D'S = 0, \quad p = F(\rho, S)$$

where $D' = \partial_t + v\partial_y + w\partial_z$.

To reduce (5.1) to involution it suffices to observe that, by the fourth equation, u_x is a function of t, y and z only. Thus, for the SF u one can take the representation

$$u = (x + X)/h \tag{5.2}$$

with some functions $X = X(t,y,z)$ and $h = h(t,y,z)$. On substituting (5.2), system (5.1) becomes

$$\rho D'v + p_y = 0, \quad \rho D'w + p_z = 0, \quad D'\rho + \rho(v_y + w_z) = -\rho / h \tag{5.3}$$
$$D'S = 0, \quad D'X = 0, \quad D'h = 1$$

which is a system in involution. It can be treated as a submodel of two-dimensional gas motions with a mass source (the first term in the third equation), which depends on the solution.

If the density and pressure are modified so that $\rho^* = h\rho$, $p^* = hp$, from (5.3) we formally obtain a submodel of two-dimensional gas motions without a source, but with the "equation of state" depending on h: $p^* = hF(\rho^*/h, S)$.

6. REGULAR SUBMODEL OF TYPE (2.2)

It is generated by the subalgebras $L_{4,47}$ and $L_{5,14}$ having the same invariants t, x, v, ρ, S and SFs u, w. The solution can be represented as

$$(u, \rho, S)|(t,x); \quad (v, w)|(t,x,y,z)$$

With the auxiliary invariant function $h = h(t,x)$, system (2.1) splits into the invariant subsystem

$$\rho(u_t + uu_x) + p_x = 0, \quad \rho_t + u\rho_x + \rho u_x + 2\rho h = 0 \tag{6.1}$$
$$S_t + uS_x = 0, \quad p = F(\rho, S)$$

and the overdetermined subsystem for the SFs v, w

$$v_t + uv_x + vv_y + wv_z = 0, \quad w_t + uw_x + vw_y + ww_z = 0 \tag{6.2}$$
$$v_y + w_z = 2h$$

Here the consistency conditions are

$$v_y w_z - v_z w_y = k(t,x) \tag{6.3}$$
$$h_t + uh_x + 2h^2 = k, \quad k_t + uk_x + 2hk = 0$$

System (6.1)–(6.3) is in involution. One can see that subsystem (6.2), (6.3) is similar to the canonical system (4.1)–(4.4) and is identical with it exactly on introducing the Lagrangian coordinate $\xi = \xi(t,x)$ as a solution of the equation $\xi_t + u\xi_x = 0$ and substituting $(t,x) \to (t, \xi)$. Thus, subsystem (6.2), (6.3) can be integrated explicitly and only subsystem (6.1) remains. It describes the one-dimensional motion of a gas with a mass source $2\rho h$.

The function $\xi(t,x)$ can be chosen in such a way that the expressions

$$\rho = k\xi_x, \quad \rho u = -k\xi_t \tag{6.4}$$

hold, which integrate the second equation in (6.1) exactly. Since $S = S(\xi)$ here, for $\xi(t,x)$ one can obtain one second-order quasilinear equation with known coefficients (by analogy with the case of steady one-dimensional gas motions).

7. REGULAR SUBMODELS OF TYPE (2.1)

According to Table 1, there are 12 such models altogether and all of them are generated by some three-dimensional subalgebras $L_{3,i}$ from the optimal system ΘL_{11} [1]. A detailed description of these submodels exists. Here it is given in an abridged form in Table 2.

In the first column in Table 2 we give the numbers i of the generating subalgebras $K_{3\,i}$. In the second column we list the operator bases of $L_{3,i}$ using the notation of (2.2), each operator X_k being represented only by its number k. The symbol $\alpha 7 + 11$, where α is any real number, denotes the operator $\alpha X_7 + X_{11}$, and so on. In the next two columns we give the bases of invariants of the subalgebras $L_{3,i}$ using the following standard notation: $r = \sqrt{(x^2 + y^2 + z^2)}$, $R = \sqrt{(y^2 + z^2)}$, $\theta = \mathrm{arctg}(z/y)$, $V = v\cos\theta + w\sin\theta$,

Table 2

i	Basis $L_{3,i}$	Invariants: Independent	Invariants: Unknown	SF	Char. class
6	$1, 4, \alpha 7 + 11$	$R/t, \theta - \alpha \ln t$	V, W	u	χ^s
8	$7, 8, 9$	t, r	U, H	ω	χ^e
11	$1, 4, 7$	t, R	V, W	u	χ^e
13	$2, 3, 7$	t, x	u, q	φ	χ^e
15^{00}	$3 + 5, 2–6, 7$	t, x	u, V^*	θ^*	χ^e
17	$1, 4, 7 + 10$	$R, \theta - t$	V, W	u	χ^s
23	$1, 4, \alpha 6 + 11$	$y/t, z/t - \alpha \ln t$	$v, w - z/t$	u	χ^s
27^{00}	$3, 6, 4 + 10$	$x - \frac{1}{2}t^2, y$	$u - t, v$	w	χ^s
29	$1, 4, 10$	y, z	v, w	u	χ^s
38_1^{000}	$3, 1 + 5, 6$	$t, x - y/t$	$u, v - y/t$	w	χ^e
38_2^{000}	$3, 5, 2 + 6$	t, x	$u, w + tv - y$	v	χ^e
46	$1, 2, 4$	t, z	v, w	u	χ^e

$W = -v \sin \theta + w \cos \theta$, $q = \sqrt{(v^2 + w^2)}$, $\varphi = \operatorname{arctg}(w / v)$. Individual symbols are adopted for $i = 8$, where U and H are, respectively, the radial component and the component tangential to the spheres $r = \text{const}$ of the velocity vector $\mathbf{u}$, and ω is the angle between the projection of $\mathbf{u}$ onto the sphere and the meridian. For $i = 15^{00}$ we introduce V^* and θ^* by

$$v = \frac{ty + z}{t^2 + 1} + V^* \cos\theta^*, \quad w = \frac{tz - y}{t^2 + 1} + V^* \sin\theta^*$$

In all submodels ρ and S are also invariants, which are omitted for brevity. In the fifth column we list the SFs. The last column presents an additional qualitative feature of the submodel, namely, its *characteristic class* χ^e or χ^s. The class χ^e contains submodels whose equations are analogous to those of *one-dimensional unsteady motions* of hyperbolic type. The class χ^s contains models whose equations are analogous to those of *two-dimensional steady flows* of special elliptic-parabolic type.

For all submodels in Table 2 we have established the existence of the corresponding partially invariant solutions. The submodel for $i = 8$ was studied in [5]. When the submodel for $i = 15^{00}$ was analysed, it turned out that reduction to an invariant solution occurs in it. All remaining submodels in Table 2 are irreducible.

8. REGULAR SUBMODELS OF TYPE (1.2)

All but one of such submodels are generated by the five-dimensional subalgebras $L_{5,i}$. The submodel generated by the subalgebra $L_{6,10}$ with basis $X_1, X_2, X_3, X_7, X_8, X_9$, invariants t, $|\mathbf{u}|$, ρ, S, and SFs u, w is an exception. It describes special barochronic motions in which the modulus of the velocity is constant

$$u^2 + v^2 + w^2 = a^2 \quad (a = \text{const}) \tag{8.1}$$

Such solutions exist, but for the corresponding overdetermined system (3.3) supplemented with (8.1) the general solution has not been found in closed form.

A list of generating subalgebras $L_{5,i}$ is given in Table 3, which is similar to Table 2. Here an important *doubling effect* is taken into account: different subalgebras generate the same submodel. Such subalgebras have the same universal invariants, which is possible because of the special form of the representation of the Lie algebra L_{11} by the operators (2.2). The doubling effect is taken into account in Table 3 by specifying all values of i that determine a given submodel along with the six-dimensional subalgebras (the last column) generating the same submodel.

The submodel for $i = 17$ has already been referred to in Section 4 as the canonical submodel. It has been established that in each submodel ($i = 10, 13, 18, 19$) there is a subsystem of equations equivalent to the canonical subsystem of type (1.2).

Table 3

i	Basis $L_{3,i}$	Invariants		SF	Doublets	
		Independent	Unknown		$r=5$	$r=6$
7	1, 5, 6, α4 + 7, β4 + 11	R/t	$u-\alpha\varphi-\beta\ln t$	q, φ		
10	2, 3, 5, 6, β4 + 7 + α11	$x/t-(\beta/\alpha)\ln t$	$u-x/t$	v, w	26	6^0
13	2, 3, 5, 6 β4 + 7	t	$u-x/t$	v, w	35	15
15	1, 2, 3, 4, 7	t	q	u, φ		
16	1, 4, 3 + 5, 2–6, 7	t	V^*	u, θ^*		
17	2, 3, 5, 6, 1 + 7	t	u	v, w	37	8^{00}
18	2, 3, 5, 6, β4 + 7 + β10	$x-\frac{1}{2}t^2$	$u-t$	v, w	31	12^{00}
19	2, 3, 5, 6, 7 + 10	x	u	v, w	33	13
36	2, 3, 4, 5, 1 + 6	t	$w-tu-x$	u, v		

9. REGULAR SUBMODELS OF TYPE (1.1)

All 29 submodels of this type are generated by the four-dimensional subalgebras $L_{4,j}$. A brief description of them is presented in Table 4, constructed in the same way and using the same notation as Tables 2 and 3. Additionally, the invariants q^*, φ^*, j_1, j_2 appearing here are given by

$$v=y/t+q^*\cos\varphi^*,\quad w=z/t+q^*\sin\varphi^*$$

$$j_1=(t^2-\alpha\beta)v+(\sigma t-\beta\tau)u-ty+\beta z$$

$$j_2=(t^2-\alpha\beta)w+(\tau t-\alpha\sigma)u-\alpha y+tz$$

For all submodels in Table 4 the corresponding $H(1,1)$-solutions exist. The submodels in which t is an independent variable give rise to partial barochronic motions of a gas.

10. CONCLUDING REMARKS

As a result of this study we have obtained a complete list of 100 regular partially invariant submodels of the equations of gas dynamics (2.1) with an equation of state of the gas of the general form $p=F(\rho, S)$. Their significance for gas dynamics is determined in the first place by the fact that they describe exact solutions of (2.1). Many of the submodels listed above are physically relevant because problems with special initial data can be posed for them.

For example, the submodel generated by the subalgebra $L_{4,48}$ in Table 4, which describes a special case of barochronic motions of a gas yields solutions of the form

$$u=0,\quad v=z-tw,\quad w=w(t,x,y,z),\quad \rho=\rho(t),\quad S=\text{const}$$

where the given functions satisfy the system of equations

$$w_t+vw_y+ww_z=0,\quad w_z-tw_y=h$$

$$\rho'/\rho=-h$$

$$h'+h^2=2k,\quad k'+hk=0,\quad p=F(\rho)$$

in which $h=h(t)$, $k=k(t)$ and a prime denotes the derivative with respect to t.

Table 4

i	Basis $L_{5,i}$	Invariants		SF
		Independent	Unknown	
1	7, 8, 9, 11	r/t	U, H	ω
4	1, 4, 10, 7 + α11	$Re^{-\alpha\theta}$	$q, \varphi-\theta$	u
5^0	5, 6, 7, β4 + 11	$x/t-\beta\ln t$	$u-x/t, q^*$	φ^*
6	1, 4, 7, 11	R/t	$q, \varphi-\theta$	u
7^0	2, 3, 7, β4 + 11	$x/t-\beta\ln t$	$u-x/t, q$	φ
9^0	1, 5, 6, β4 + 7	t	$u-\beta\varphi^*, q^*$	φ^*
10^0	2, 3, 4, 7	t	$u-x/t, q$	φ
12	1, 2, 3, β4 + 7	t	$u-\beta\varphi, q$	φ
13	7, 8, 9, 10	r	U, H	ω
14	2, 3, 7, 10	x	u, q	φ
16^0	2, 3, 7, 4 + 10	$x-\frac{1}{2}t^2$	$u-t, q$	φ
17	4, 5, 6, 7	t	$u-x/t, q^*$	φ^*
18	4, 5, 6, 1 + 7	t	$u+(\varphi^*-x)/t, q^*$	φ^*
19	4, 3 + 5, 2–6, α1 + 7	t	$u+(\alpha\theta^*-x)/t, V^*$	θ^*
20	1, 3 + 5, 2–6, α4 + 7	t	$u-\alpha\theta^*, V^*$	θ^*
21	2, 3, 4, 1 + 7	t	$u+(\varphi-x)/t, q$	φ
23	1, 4, 10, 11	z/y	v, w	u
29	1, 4, 6, α5 + 11	$y/t-\alpha\ln t$	$v-y/t,\ w-z/t$	u
30^0	2, 3, 6, β4 + σ5 + 11	$x/t-\beta\ln t$	$u-x/t,\ v-\sigma\ln t$	w
35^0	2, 3, 5, 4 + β6 + 10	$x-\frac{1}{2}t^2$	$u-t,\ w-\beta t$	v
36^0	2, 3, 5, 6 + 10	x	$u, w-t$	v
38	2, 3, 5, 10	x	u, w	v
41	1, σ2 + τ3 + 4, α3 + 5, β2 + 6	t	$j_1,\ j_2$	u
42	1, 4, 3 + 5, 2–6	t	$V^*,\ \theta^*$	u
43	1, 4, 5, 6	t	$v-y/t,\ w-z/t$	u
44	2, α1 + 3, 1 + 5, 6	t	$u, v-\alpha tw-x+\alpha z$	w
46	2, α1 + 3, 5, 6	t	$u, w+(x-\alpha z)/\alpha t$	v
48	1, 2, 3 + 5, 6	t	$u, v+tw-z$	w
50	1, 2, 3, 4	t	v, w	u

The solution of this system is uniquely defined by the initial data for $t = 0$

$$u=0, \quad v=z, \quad w=-k_0y+h_0z+W(x)$$
$$\rho=\rho_0, \quad h=h_0, \quad k=k_0$$

where ρ_0, k_0, h_0 are arbitrary constants and $W(x)$ is an arbitrary function. The solution can be found explicitly and contains an arbitrary function of one argument. It describes the motion of a gas as a type of wave propagating over an expanding spatial background.

For special equations of state (a polytropic gas, etc.) such a list will be, in general, larger in accordance with the classification of "large" models of gas dynamics [1].

There are also many irregular models for system (2.1), but they remain to be investigated in the future. Such submodels exist, as a rule, only for special equations of state. The problems of their existence is related to non-trivial issues on reducing overdetermined systems to involution. One can get some idea of the difficulties arising in this case from [6, 7], where (among other things) the problem is discussed for partially invariant submodels of type (2.2) and (3.3) generated by the subalgebra $L_{4,40}$ with basis X_1, X_2, X_3, X_{10}.

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